

# Equidistribution Rates, Closed String Amplitudes, and the Riemann Hypothesis.

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**ABSTRACT:** We study asymptotic relations connecting unipotent averages of  $Sp(2g, \mathbb{Z})$  automorphic forms to their integrals over the moduli space of principally polarized abelian varieties. We obtain reformulations of the Riemann hypothesis as a class of problems concerning the computation of the equidistribution convergence rate in those asymptotic relations. We discuss applications of our results to closed string amplitudes. Remarkably, the Riemann hypothesis can be rephrased in terms of ultraviolet relations occurring in perturbative closed string theory.

**KEYWORDS:** String Theory, Equidistribution, Unipotent Flows, Automorphic Forms, Riemann hypothesis.

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## 1. Introduction

It is well known that long horocycles tend to become uniformly distributed in the unit tangent bundle of the modular surface  $Sl(2, \mathbb{Z}) \backslash Sl(2, \mathbb{R})$ , [1],[2]. For a large class of homogenous spaces given by quotients of Lie groups by discrete subgroups, Ratner theorems on equidistribution of unipotent flows [3],[4] provide the key for proving striking results in number theory. Thanks to equidistribution, the unipotent average of an automorphic function  $f$  in a suitable limit converges to its average on the homogenous space. A very interesting quantity is the  $f$  convergence rate for its connection to the (Grand)Riemann hypothesis. This quantity is not provided by Ratner algebraic methods. For the  $Sl(2, \mathbb{Z}) \backslash Sl(2, \mathbb{R})$  modular surface, Zagier [5], (see also [6],[7], [8] as related works) has shown that if just in a single case of  $Sl(2, \mathbb{Z})$ -invariant smooth function  $f$  of rapid decay as  $z \rightarrow i\infty$  the horocycle convergence rate is  $O(\Im(z)^{3/4})$  as  $\Im(z) \rightarrow 0$  then the Riemann hypothesis is true! Information contained in the modular surface  $Sl(2, \mathbb{Z}) \backslash Sl(2, \mathbb{R})$  on the Riemann zeta function  $\zeta(s)$  are due to the arithmetic nature of this surface, being the space of inequivalent unimodular lattices on the plane  $\mathbb{R}^2$ .

In this paper we consider  $Sp(2g, \mathbb{Z})$  automorphic forms  $f$  defined on the genus  $g$  Siegel upper space  $\mathbb{H}_g$ . By extending old ideas of Rankin [9] and Selberg [10], we study by analytic methods equidistribution of the  $f$  average along unipotent flows. A great simplification follows by the use of Iwasawa decomposition for  $Sp(2g, \mathbb{R})$  in the unfolding of the modular domain  $Sp(2g, \mathbb{Z}) \backslash Sp(2g, \mathbb{R})$ . By using analytic properties of genus  $g$  Eisenstein series associated to various components of the boundary of the modular domain, we obtain that certain values of the convergence rates of the unipotent averages are compatible only with the Riemann hypothesis. Indeed, our results provide reformulations of the

Riemann hypothesis in terms of a class of problems for the convergence rate in the unipotent dynamics in the  $Sp(2g, \mathbb{Z}) \backslash \mathbb{H}_g$  modular space. These results generalize to modular domains  $Sp(2g, \mathbb{Z}) \backslash Sp(2g, \mathbb{R})$  the  $O(\Im(z)^{3/4})$  result [5] for the genus  $g = 1$  modular surface  $Sl(2, \mathbb{Z}) \backslash Sl(2, \mathbb{R})$ .

Among the motivations for our work, there is a certain relevance of equidistribution properties of unipotent flows in String Theory [11]. Indeed, uniform distribution is behind some deep UV/IR properties of perturbative String Theory. Equidistribution of large horocycles in  $Sl(2, \mathbb{Z}) \backslash Sl(2, \mathbb{R})$  is at the heart [11] of the UV/IR connection among asymptotic Fermi-Bose degeneracy and space-time stability [12]. In a different direction, equidistribution of large horocycles involving congruence subgroups  $\Gamma_0[N] \subset Sl(2, \mathbb{Z})$  modular domains allows to write a  $\mathbb{Z}_N$  orbifold torus amplitude as a certain limit of a one-dimensional integral [11]. This may lead to some simplifications for computing one loop torus integrals for  $\mathbb{Z}_N$  orbifolds.

We find very appealing the fact that in some known example the equidistribution convergence rate corresponds to some quantity in String Theory. As an example, in [13] the genus  $g = 1$  case is studied for non tachyonic closed string backgrounds admitting a CFT description. In those cases the supertrace over the closed string states is a  $Sl(2, \mathbb{Z})$  automorphic function, and the horocycle average of the automorphic supertrace counts the difference among the total number of closed strings bosonic minus fermionic excitations below an ultraviolet cutoff controlled by the horocycle radius.

Indeed, a very interesting direction which motivates our work is the search of examples in String theory where, besides the correspondence of the equidistribution rate to some “physical” quantity, also dualities which allow to map the rate are available. Then one may attempt to leave modularity by using a string duality and attempt to estimate the equidistribution rate from the other side of the duality. This latest possibility seems quite appealing and it is yet unexplored.

In the last part of this paper we briefly outline various possible applications to String Theory of our  $Sp(2g, \mathbb{R})$  results. Some of those applications are presently under investigation and will be presented in future publications. Our results allow to re-express genus  $g$  closed string amplitudes as lower dimensional integral along unipotent flows. This has interesting applications for  $g = 2$  closed string amplitudes [14],[15],[16],[17],[18],[19], and in recently proposed expression for genus  $g \geq 3$  closed string amplitudes [20],[21],[22],[23],[24] [25]. The main idea is to use equidistribution results in order to obtain constraints deriving from finiteness of genus  $g$  closed string amplitudes. This would give a genus  $g$  generalizations of the genus one asymptotic supersymmetry constraint for non tachyonic closed string spectra originally obtained in [12].

## 2. Equidistribution theorems

The genus  $g$  Siegel upper space  $\mathbb{H}_g \subset Mat(g, \mathbb{C})$  is the set of complex  $g \times g$  symmetric matrices with positive definite imaginary part  $\mathbb{H}_g = \{\tau \in Mat(g, \mathbb{C}) | \tau = \tau^t, \Im(\tau) > 0\}$ .  $\mathbb{H}_g$

is isomorphic to the Lie coset  $\mathbb{H}_g \simeq Sp(2g, \mathbb{R}) / (Sp(2g, \mathbb{R}) \cap SO(2g, \mathbb{R}))$ . For  $m$  in the coset

$$m = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad (2.1)$$

the bijective map is given by

$$\tau(m) = (ai\mathbb{I} + b)(ci\mathbb{I} + d)^{-1}. \quad (2.2)$$

The Iwasawa decomposition allows to write a symplectic matrix  $g$  in  $Sp(2g, \mathbb{R})$  as  $g = nak$ ,  $k \in SO(2g, \mathbb{R}) \cap Sp(2g, \mathbb{R})$ ,  $a$  is a positive definite diagonal matrix and  $n$  is a unipotent matrix. It is convenient to employ the following  $g \times g$  blocks parametrization

$$a = \begin{pmatrix} V & 0 \\ 0 & V^{-1} \end{pmatrix}, \quad V = \text{diag}(\sqrt{v_1}, \dots, \sqrt{v_g}), \quad (2.3)$$

for the Abelian part with  $v_i > 0$ ,  $i = 1, \dots, g$ , and

$$n = \begin{pmatrix} U & WU^{-t} \\ 0 & U^{-t} \end{pmatrix} \quad (2.4)$$

for the unipotent part, with  $W$  symmetric real  $g \times g$  matrix

$$W = \begin{pmatrix} w_{11} & w_{12} & \dots & w_{1g} \\ w_{12} & w_{22} & \dots & w_{2g} \\ \vdots & \vdots & \ddots & \vdots \\ w_{1g} & w_{2g} & \dots & w_{gg} \end{pmatrix}$$

and  $U$  upper triangular real  $g \times g$  matrix

$$U = \begin{pmatrix} 1 & u_{12} & \dots & u_{1g} \\ 0 & 1 & \dots & u_{2g} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

With the above parametrization,  $\mathbb{H}_g$  in Iwasawa coordinates is given by (2.2)

$$\tau(m) = W + iUV^2U^t. \quad (2.5)$$

We are interested in  $\Gamma_g \sim Sp(2g, \mathbb{Z})$  automorphic forms, and in particular in proving equidistribution of subgroup flows in order to reduce a typical modular integral

$$\int_{\Gamma_g \backslash \mathbb{H}_g} d\mu f(\tau), \quad (2.6)$$

into a lower dimensional integral.  $f$  is a weight zero automorphic function under the modular group  $\Gamma_g \sim Sp(2g, \mathbb{Z})$ ,  $f(\gamma(\tau)) = f(\tau)$ ,  $\gamma \in \Gamma_g$ , and  $d\mu$  is the  $\Gamma_g$ -invariant hyperbolic  $\mathbb{H}_g$  measure

$$d\mu = \frac{1}{\det(\Im(\tau))^{g+1}} \prod_{i \leq j} d\Re(\tau)_{ij} d\Im(\tau)_{ij}, \quad (2.7)$$

where  $\tau_{ij} = \Re(\tau)_{ij} + i\Im(\tau)_{ij}$ .

The general strategy will be to consider the following modular integral

$$I_{g,r}(s) = \int_{\Gamma_g \backslash \mathbb{H}_g} d\mu f(\tau) \phi_r(\tau, s), \quad (2.8)$$

where  $\phi_r(\tau, s)$  is a suitable genus  $g$  Poincaré series. An interesting family of Poincaré series has the form

$$\phi_r(\tau, s) = \sum_{\Gamma_g \cap P_{g-r} \backslash \Gamma_g} \varphi \left( \frac{\det(\Im(\gamma(\tau)))}{\det(\Im((\gamma(\tau)_{22}))), s} \right), \quad (2.9)$$

where we use the following block decomposition

$$\tau = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12}^t & \tau_{22} \end{pmatrix}, \quad (2.10)$$

with  $\tau_{11} \in \mathbb{H}_r$  and  $\tau_{22} \in \mathbb{H}_{g-r}$ ,  $1 \leq r \leq g-1$ .  $P_{g-r} \subset Sp(2g, \mathbb{R})$  is the parabolic subgroup which stabilizes the  $(g-r)$ -dimensional (in the complex sense) rational component of the boundary of  $\Gamma_g \backslash \mathbb{H}_g$ , (see the appendix for an account on the properties of the  $\Gamma_g \backslash \mathbb{H}_g$  boundary and related parabolic subgroups of  $Sp(2, \mathbb{R})$ ).

From (2.5) it follows that

$$\det(\Im(\tau)) = \prod_{i=1}^g v_i, \quad (2.11)$$

and therefore the argument of the Poincaré series is given by

$$\frac{\det(\Im(\tau))}{\det(\Im(\tau_{22}))} = \prod_{i=1}^r v_i. \quad (2.12)$$

This means that  $\phi_r(\tau, s)$  is constructed by summing over the images under the modular group  $\Gamma_g$  of a function  $\varphi(v^{(r)}, s)$  of  $U \backslash \mathbb{H}_r$ , where  $v^{(r)} = (v_1, \dots, v_r)$ .

If some of the Abelian coordinates  $v_i \rightarrow 0$ , ( $i = 1, \dots, r$ ),  $\tau$  reaches the  $(g-r)$ -dimensional component of the boundary of  $\mathbb{H}_g$ , which is stabilized by the parabolic subgroup  $P_{g-r}$ . The left quotient by  $P_{g-r} \cap \Gamma_g$  in eq. (2.9) avoids overcounting modular transformations, as  $\tau$  reaches the  $(g-r)$ -dimensional component of the  $\mathbb{H}_g$  boundary.

Under suitable boundary conditions at the  $r$ -dimensional component of the  $\Gamma_g \backslash \mathbb{H}_g$  boundary for the modular invariant function  $f(\tau)$ , the modular integral  $I_{g,r}(s)$  inherits the

$\phi_r(\tau, s)$  analytic properties in the complex variable  $s$ . In Iwasawa coordinates, the modular integral acquires the following form

$$I_{g,r}(s) = \int_{\Gamma_g \backslash \mathbb{H}_g} d\vec{w} d\vec{u} \prod_{i=1}^g dv_i v_i^{i-g-2} f(\tau) \phi_r(\tau, s), \quad (2.13)$$

where the Jacobian determinant  $J$  of the transformation (2.5) is given by

$$J = \prod_{i=1}^g v_i^{i-1}. \quad (2.14)$$

By using the unfolding trick, eq. (2.13) reduces to

$$\begin{aligned} I_{g,r}(s) &= \int_{(P_{g-r} \cap \Gamma_g) \backslash \mathbb{H}_g} d\vec{w} d\vec{u} \prod_{i=1}^g dv_i v_i^{i-g-2} f(\vec{v}, \vec{w}, \vec{u}) \varphi_r(v^{(r)}, s) \\ &= \int_0^\infty dv_1 \dots \int_0^\infty dv_g \prod_{i=1}^g v_i^{i-g-2} \varphi_r(v^{(r)}, s) \int_{(P_{g-r} \cap \Gamma_g) \backslash U} d\vec{w} d\vec{u} f(\vec{v}, \vec{w}, \vec{u}) \\ &= \int_0^\infty dv_1 \dots \int_0^\infty dv_g \prod_{i=1}^g v_i^{i-g-2} \varphi_r(v^{(r)}, s) \langle f \rangle_{(P_{g-r} \cap \Gamma_g) \backslash U}(\vec{v}) \end{aligned} \quad (2.15)$$

$\langle f \rangle_{(P_r \cap \Gamma_g) \backslash U}(\vec{v})$  is the  $f$  average along the unipotent flow computed with the  $\Gamma_g$ -invariant metric

$$\langle f \rangle_{(P_{g-r} \cap \Gamma_g) \backslash U}(\vec{v}) =: \int_{(P_{g-r} \cap \Gamma_g) \backslash U} d\vec{w} d\vec{u} f(\vec{v}, \vec{w}, \vec{u}). \quad (2.16)$$

It is interesting to check whether the meromorphic structure of  $I_{g,r}(s)$  constrains (some)  $v_i \rightarrow 0$  limits of the unipotent average of the modular invariant  $f$ .

We start by studying the modular integral of the product of a  $Sp(2g, \mathbb{Z})$  invariant function  $f(\tau)$  times the  $E_{g,r}(\tau, s)$  Eisenstein series

$$I_{g,r}(s) = \int_{\Gamma_g \backslash \mathbb{H}_g} d\mu f(\tau) E_r(\tau, s), \quad (2.17)$$

$$E_{g,r}(\tau, s) = \sum_{\Gamma_g \cap P_{g-r} \backslash \Gamma_g} \left( \frac{\det(\Im(\gamma(\tau)))}{\det(\Im((\gamma(\tau)_{22})))} \right)^s. \quad (2.18)$$

Analytic properties of the Eisenstein series  $E_{g,r}(\tau, s)$  are given for example in [26] (theorem 2.2 and theorem 2.3). Following [26], the genus  $g$  dressed Eisenstein series  $\mathcal{E}_{g,r}(\tau, s)$ ,  $r = 2, \dots, g-1$

$$\mathcal{E}_{g,r}(\tau, s) = \prod_{i=1}^r \zeta^*(2s+1-i) \prod_{i=1}^{[r/2]} \zeta^*(4s-2g+2r-2i) E_{g,r}(\tau, s) \quad (2.19)$$

is meromorphic in the complex variable  $s$ , with a simple pole in  $s = g - (r-1)/2$  with residue

$$\frac{1}{2} \prod_{j=2}^r \zeta^*(j) \prod_{j=1}^{[r/2]} \zeta^*(2g-2r+2j+1),$$

where  $\zeta^*(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ , and  $[x]$  denotes the integral part of the real number  $x$ . Analytic properties of  $\mathcal{E}_{g,r}(\tau, s)$  and eq. (2.19) imply that  $E_{g,r}(\tau, s)$  is meromorphic on the  $s$  plane with a simple pole in  $s = g - (r-1)/2$  with residue

$$Res_{s \rightarrow g-(r-1)/2} E_{g,r}(\tau, s) = \frac{\frac{1}{2} \prod_{j=2}^r \zeta^*(j) \prod_{j=1}^{[r/2]} \zeta^*(2g-2r+2j+1)}{\prod_{i=1}^r \zeta^*(2g-r+2-i) \prod_{i=1}^{[r/2]} \zeta^*(2g+2-2i)}. \quad (2.20)$$

Moreover,  $E_{g,r}(\tau, s)$  has poles in  $s = \frac{\rho}{2} + \frac{i-1}{2}$  and  $s = \frac{\rho}{4} + \frac{g-r+i}{2}$ , for  $i = 1, \dots, r$ , where  $\rho$  are the non trivial zeros of the Riemann zeta function,  $\zeta^*(\rho) = 0$ .

In the  $r = 1$  case,  $E_{g,1}(\tau, s)$  has a simple pole in  $s = g$  with residue  $1/\zeta^*(2g)$  and poles in  $\rho/2$ , where  $\rho$ 's are the non trivial zeros of the Riemann zeta function, ([26], theorem 2.3). After the unfolding trick illustrated in (2.15), the modular integral (2.17) reduces to

$$I_{g,r}(s) = \int_{(\Gamma_g \cap P_{g-r}) \setminus \mathbb{H}_g} dv_i v_i^{i-g-2+s} \prod_{j=r+1}^g dv_j v_j^{j-g-2} d\vec{u} d\vec{w} f(\vec{v}, \vec{u}, \vec{w}). \quad (2.21)$$

## 2.1 Modular holography, equidistribution convergence rates and the Riemann hypothesis

In this section we develop a method for reducing a given modular integral of a  $Sp(2g, \mathbb{Z})$ -invariant function  $f(\tau)$  over a fundamental domain  $\Gamma_g \setminus \mathbb{H}_g$  into an integral of the  $f(\tau)$  average along some unipotent directions, over the  $(g-1)$ -dimensional component  $F_{g-1}$  of the boundary of  $\mathbb{H}_g$ . By iterating this method one can then reduce the original genus  $g$  integral into the limit towards the zero-dimensional component  $F_0$  of the  $\mathbb{H}_g$  boundary of the average of  $f(\tau)$  along all the unipotent directions in  $\mathbb{H}_g$ . Moreover, we will prove that equidistribution convergence rates for the various lower dimensional integrals we obtain in the process are related to the Riemann hypothesis.

The  $(g-1)$ -component  $F_{g-1}$  of the  $\Gamma_g \setminus \mathbb{H}_g$  boundary is given by

$$F_{g-1} = \begin{pmatrix} i\infty & 0 \\ 0 & \tau_{g-1} \end{pmatrix}, \quad \tau_{g-1} \in \mathbb{H}_{g-1}. \quad (2.22)$$

In Iwasawa coordinates  $\mathbb{H}_g$  is given by

$$\tau = W + iUV^2U^t, \quad (2.23)$$

and with the convention for  $U$  to be a upper triangular matrix, one finds for the first  $\tau$  entry

$$\tau_{11} = w_{11} + i(v_1 + \bar{u}V_{g-1}^2\bar{u}^t), \quad (2.24)$$

where  $\bar{u}$  is the  $(g-1)$ -dimensional row vector with components  $\bar{u} = (u_{12}, \dots, u_{1g})$  and  $V_{g-1}^2 = \text{diag}\{v_2, \dots, v_g\}$ .

For  $r = 1$ , the unfolding equation (2.21) gives

$$\int_{\Gamma_g \setminus \mathbb{H}_g} d\mu f(\tau) E_{g,1}(\tau, s) = \int_{(\Gamma_g \cap P_{g-1}) \setminus \mathbb{H}_g} dv_1 v_1^{s-g-1} \prod_{j=2}^g dv_j v_j^{j-g-2} d\vec{u} d\vec{w} f(\vec{v}, \vec{u}, \vec{w}), \quad (2.25)$$

where the  $P_{g-1} \cap \Gamma_g$  parabolic subgroup which stabilizes the rational component  $F_{g-1}$  of the  $\mathbb{H}_g$  boundary is given by the following matrices

$$\begin{pmatrix} 1 & * & * & * \\ * & a & 0 & b \\ 0 & 0 & 1 & * \\ * & c & 0 & d \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{g-1}. \quad (2.26)$$

For the generic matrix in  $P_{g-1}$  one has the decomposition

$$\begin{pmatrix} 1 & m & q & n \\ 0 & a & n^t & b \\ 0 & 0 & 1 & 0 \\ 0 & c & -m^t & d \end{pmatrix} = g_1 \cdot g_2 \cdot g_3, \quad (2.27)$$

with, (see for example [27]),

$$g_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & d \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{g-1}, \quad (2.28)$$

$$g_2 = \begin{pmatrix} 1 & m & 0 & n \\ 0 & 1 & n^t & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -m^t & 1 \end{pmatrix} \quad m, n \in \text{Mat}(1 \times (g-1), \mathbb{Z}), \quad (2.29)$$

and

$$g_3 = \begin{pmatrix} 1 & 0 & q & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad q \in \mathbb{Z}. \quad (2.30)$$

On  $\tau \in \mathbb{H}_g$

$$\tau = \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2^t & \tau_3 \end{pmatrix} \quad \tau_1 \in \mathbb{H}_1, \tau_2 \in \text{Mat}(1 \times (g-1), \mathbb{C}), \tau_3 \in \mathbb{H}_{g-1}, \quad (2.31)$$

the action of  $g_1$ ,  $g_2$  and  $g_3$  is given by [27]

$$g_1(\tau) = \begin{pmatrix} \tau_1 - \tau_2(c\tau_3 + d)^{-1}c\tau_2^t & * \\ (c\tau_3 + d)^{-1}\tau_2^t & (a\tau_3 + b)(c\tau_3 + d)^{-1} \end{pmatrix}, \quad (2.32)$$

$$g_2(\tau) = \begin{pmatrix} \tau_1' & * \\ \tau_2^t + m^t\tau_1 + n^t & \tau_3 \end{pmatrix}, \quad \tau_1' = \tau_1 + m\tau_3m^t + m^t\tau_2 + (m^t\tau_2)^t + nm^t, \quad (2.33)$$

$$g_3(\tau) = \begin{pmatrix} \tau_1 + q & * \\ \tau_2^t & \tau_3 \end{pmatrix}, \quad (2.34)$$

where the entries  $*$  are given by symmetry of  $\tau$ .

The above decomposition shows that  $P_{g-1} \cap \Gamma_g$  contains  $\Gamma_{g-1}$  as a subgroup acting on  $\tau_3 \in \mathbb{H}_g$ . Therefore, one in eq. (2.25) can take the average of  $f(\tau)$  along the appropriate unipotent directions, in order to obtain a  $\Gamma_{g-1}$ -invariant function  $\langle f \rangle_{2g-1}(v_1, \tau_3)$ , with  $\tau_3 \in \mathbb{H}_{g-1}$ . This can be obtained as follows. Specializing to the case  $r = 1$ , it is convenient to express the Iwasawa parametrization of  $\tau$  by evidencing the  $(g-1)$ -structures:

$$U =: \begin{pmatrix} 1 & \bar{u} \\ \bar{0}^t & U_{g-1} \end{pmatrix}, \quad V^2 = \text{diag}\{v_1; \bar{v}\}, \quad (2.35)$$

where  $\bar{u}$  is the  $(g-1)$ -dimensional row vector  $\bar{u} = (u_{12}, \dots, u_{1g})$ , and  $\bar{v} = (v_2, \dots, v_g)$ . Then (2.5) takes the form

$$\tau = \tau_{g-1} + \begin{pmatrix} w_{11} & \bar{w} \\ \bar{w}^t & \mathbb{O} \end{pmatrix} + i \begin{pmatrix} v_1 + \bar{u}V_{g-1}^2\bar{u}^t & \bar{u}V_{g-1}^2U_{g-1}^t \\ U_{g-1}V_{g-1}^2\bar{u}^t & \mathbb{O} \end{pmatrix}, \quad (2.36)$$

where

$$\tau_{g-1} = \begin{pmatrix} 0 & \bar{0} \\ \bar{0}^t & \tau_3 \end{pmatrix}, \quad (2.37)$$

$\bar{w} = (w_{12}, \dots, w_{1g})$   $U_{g-1}$  is the minor of  $U_{11}$ , and  $\mathbb{O}$  is the null squared  $(g-1)$ -dimensional matrix. Using this in eq. (2.25) one gets

$$\int_{\Gamma_g \backslash \mathbb{H}_g} d\mu_g f(\tau) E_{g,1}(\tau, s) = \int_0^\infty dv_1 v_1^{s-g-1} \int_{\Gamma_{g-1} \backslash \mathbb{H}_{g-1}} d\mu_{g-1} \langle f \rangle_{2g-1}(v_1, \tau_3), \quad (2.38)$$

where in the average  $\langle f \rangle_{2g-1}(v_1, \tau_3)$  the integration over the  $(2g-1)$  unipotent coordinates  $w_{11}, \bar{u}, \bar{w}$  that reduce  $\mathbb{H}_g \rightarrow \mathbb{H}_{g-1}$  takes into account the identifications induced by the left quotient by the parabolic subgroup  $P_{g-1}$ .

In the r.h.s. of eq. (2.38) the Mellin transform of the following function appears

$$\mathcal{F}_{g-1}(v_1) =: \frac{1}{v_1^g} \int_{\Gamma_{g-1} \backslash \mathbb{H}_{g-1}} d\mu_{g-1} \langle f \rangle_{2g-1}(v_1, \tau_3). \quad (2.39)$$

The following condition

$$\lim_{v_1 \rightarrow 0} \int_{\Gamma_{g-1} \backslash \mathbb{H}_{g-1}} d\mu_{g-1} \langle f \rangle_{2g-1}(v_1, \tau_3) = C_g, \quad (2.40)$$

reproduces the simple pole in  $s = g$  of the genus  $g$ ,  $r = 1$  Eisenstein series  $E_{g,1}(\tau, s)$ , whose analytic properties are given before the end of the previous section. One has

$$\text{Res}_{s \rightarrow g} E_{g,1}(\tau, s) = \frac{1}{2\zeta^*(2g)} = \frac{\text{Vol}(\Gamma_{g-1} \backslash \mathbb{H}_{g-1})}{2\text{Vol}(\Gamma_g \backslash \mathbb{H}_g)}, \quad (2.41)$$

since

$$\text{Vol}(\Gamma_g \backslash \mathbb{H}_g) = 2 \prod_{k=1}^g \zeta^*(2k). \quad (2.42)$$

Therefore one has in eq. (2.38) as  $v_1 \rightarrow 0$

$$\int_{\Gamma_{g-1} \setminus \mathbb{H}_{g-1}} d\mu_{g-1} < f >_{2g-1} (v_1, \tau_3) \sim \frac{\text{Vol}(\Gamma_{g-1} \setminus \mathbb{H}_{g-1})}{\text{Vol}(\Gamma_g \setminus \mathbb{H}_g)} \int_{\Gamma_g \setminus \mathbb{H}_g} d\mu_g f(\tau) \quad v_1 \rightarrow 0 \quad (2.43)$$

We can also obtain the  $v_1 \rightarrow 0$  convergence rate in (2.43), which is quite interestingly related to the Riemann hypothesis. Due to the location of the poles of  $E_{g,1}(\tau, s)$ , the Mellin transform in the r.h.s. of eq. (2.38) is analytic on the half-plane  $\Re(s) > \frac{\Theta}{2}$ , except for a simple pole in  $s = g$ .  $\Theta = \text{Sup}\{\Re(\rho) | \zeta^*(\rho) = 0\}$  is the superior of the real part of the non trivial zeros of the Riemann zeta function. By inverse Mellin transform argument, one then finds the following  $v_1 \rightarrow 0$  error estimate

$$\int_{\Gamma_{g-1} \setminus \mathbb{H}_{g-1}} d\mu_{g-1} < f >_{2g-1} (v_1, \tau_3) \sim \frac{\text{Vol}(\Gamma_{g-1} \setminus \mathbb{H}_{g-1})}{\text{Vol}(\Gamma_g \setminus \mathbb{H}_g)} \int_{\Gamma_g \setminus \mathbb{H}_g} d\mu_g f(\tau) + O(v_1^{g-\frac{\Theta}{2}}). \quad (2.44)$$

Notice that the above convergence rate is  $O(v_1^{g-\frac{1}{4}})$  iff the Riemann hypothesis is true! This result for the  $Sp(2g, \mathbb{Z})$  equidistribution rate corresponds to the  $3/4$  rate condition [5] in the  $Sl(2, \mathbb{Z})$  case for functions of rapid decay  $z \rightarrow i\infty$ , which is indeed recovered in (2.44) by setting  $g = 1$ .

By iteration of the above method it looks that one recovers equidistribution of the  $f(\tau)$  average over the full set of unipotent coordinates of  $\mathbb{H}_g$  in the zero-dimensional boundary limit. Furthermore, one obtains the convergence rates for the  $\mathbb{H}_g$  abelian coordinates, and an intriguing connection among the values of the powers of the error terms and the Riemann hypothesis. A computation of those values which do not beg on the Riemann zeta function would prove or disprove the Riemann hypothesis. It would be therefore quite interesting to be able to map the problem in String Theory terms and gain a new angle from which to estimate those quantities.

### 3. Conclusions and perspectives

In this paper we have started to tackle the problem of determine equidistribution properties for genus  $g \geq 2$   $Sp(2g, \mathbb{Z})$  automorphic forms defined on the Siegel upper space  $\mathbb{H}_g$ . Our main aim is to explore the potential applications to genus  $g \geq 2$  string amplitudes. For  $g = 1$  the relevance of uniform distribution has been put in evidence in [11], where it is shown how to determine constraints on the ultraviolet property of non tachyonic closed string spectra [12].

For genus  $g = 2$ , the vacuum-to-vacuum superstring amplitudes has been computed in [14]–[19], and recent proposals for higher genus can be found, for example, in [20]–[24]. Then, it should be interesting to apply the uniformization method in order to obtain constraints related to finiteness of genus  $g$  closed string amplitudes. This is indeed under study at the moment. However, note that at the actual stage the uniformization methods can be applied to string theory only up to genus three. Indeed, (super)string integrals must be performed over the moduli space of Riemann surfaces, and for genus higher than

$g = 3$  this task is related to the Schottky problem! This suggests that it should be really interesting to try to find equidistribution theorems valid for automorphic forms defined only over the Schottky locus.

Beyond physical applications, it is interesting to note the relations with number theory: [5] has shown that the equidistribution rate is intimately related to the Riemann hypothesis, which is proven to hold true if one is able to show that for a certain class of  $Sl(2, \mathbb{Z})$  automorphic functions their horocycle convergence rate is  $O(\Im(z)^{3/4})$ . Here we have found that in the case of codimension 1 reductions, the same holds true for the  $Sp(2g, \mathbb{Z})$  case with a convergence rate  $O(\Im(z)^{(4g-1)/4})$ . To our knowledge this result is new. This reduction can be in principle reiterated to obtain higher codimension reductions. However, it is interesting to note that if one considers directly  $r$ -codimensional reductions, with  $r > 1$ , then new poles appear, which could provide new information. Thus, one should try to extend these results to such general cases.

The physical and mathematical interests can be further interweaved by the following remark: There are known examples where the convergence rate corresponds to physical quantities, see [13], for example. This suggests to investigate such correspondence more deeply in order to relate equidistribution rates to string quantities which, after dualities, could be put in a calculable form. This could provide a way to gain a string theory perspective on the Riemann hypothesis.

All these points are currently under investigation.

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### A. Appendix

#### A.1 On $\mathbb{H}_g$ boundary components and their parabolic subgroups stabilizers

In this section we review the structure of the boundary of  $\mathbb{H}_g$ . We will show that  $\partial\mathbb{H}_g$  is given by components of various dimensionality which are stabilized by parabolic subgroups of  $Sp(2g, \mathbb{R})$ . The strategy in order to study the structure of  $\partial\mathbb{H}_g$  [27] is to use the Cayley map to go from  $\mathbb{H}_g$  to the multidimensional generalization of the open unit Poincaré disc  $\mathbb{D}_g = \{z \in Mat(2g, \mathbb{C}) | z = z^t, z\bar{z} < \mathbb{I}\}$ . One then takes the closure  $\bar{\mathbb{D}}_g$ , and study the boundary of  $\Gamma_g \backslash \bar{\mathbb{D}}_g$ . This is done by constructing an equivariant map under  $Sp(2g, \mathbb{R})$  which relates boundary components of  $\Gamma_g \backslash \bar{\mathbb{D}}_g$  to isotropic subspaces in  $\mathbb{R}^{2g}$ . Since isotropic subspaces in  $\mathbb{R}^{2g}$  are stabilized by parabolic subgroups of  $Sp(2g, \mathbb{R})$ , it follows that boundary components of  $\Gamma_g \backslash \bar{\mathbb{D}}_g$  are stabilized by parabolic elements of  $Sp(2g, \mathbb{R})$ . This constructions

provides an explicit characterization of the boundary components of  $\Gamma_g \backslash \bar{\mathbb{D}}_g$  and for each component the explicit form of the parabolic subgroup stabilizer.

The Cayley map is defined as  $z : \mathbb{H}_g \rightarrow \mathbb{D}_g$

$$z(\tau) = (\tau - i\mathbb{I})(\tau + i\mathbb{I})^{-1}, \quad (\text{A.1})$$

where the bounded domain  $\mathbb{D}_g = \{z \in \text{Sym}(2g, \mathbb{C}) \mid z\bar{z} < \mathbb{I}\}$  is the multidimensional analogous of the Poincaré open disc. Let  $\bar{\mathbb{D}}_g = \{z \in \text{Sym}(2g, \mathbb{C}) \mid z\bar{z} \leq \mathbb{I}\}$  the closure of  $\mathbb{D}_g$  and  $\partial\bar{\mathbb{D}}_g = \bar{\mathbb{D}}_g - \mathbb{D}_g$  the boundary of  $\mathbb{D}_g$ .

We now define a map which allows to explore the structure of the boundary of  $\Gamma_g \backslash \bar{\mathbb{D}}_g$ . It is defined as  $\Psi_z(w) : \mathbb{C}^g \rightarrow \mathbb{C}^g$ ,  $\Psi_z(w) =: i(wz + \bar{w})$ ,  $z \in \bar{\mathbb{D}}_g$ .  $\Psi_z(w)$  enjoys the following property, let  $U(z) =: \text{Ker}(\Psi_z(w)) = \{w \in \mathbb{C}^g \mid \bar{w} = -wz\}$ , then  $U(z) \neq 0$  iff  $z \in \partial\bar{\mathbb{D}}_g$ . Moreover, the image of  $U(z)$  in  $\mathbb{R}^{2g}$  through the map  $\nu : \mathbb{C}^g \rightarrow \mathbb{R}^{2g}$ ,  $\nu_i = \frac{1}{2}(w_i + \bar{w}_i)$ ,  $\nu_{g+i} = \frac{1}{2i}(w_i - \bar{w}_i)$ ,  $i = 1, \dots, g$ , is an isotropic space, (i.e.  $\langle \nu, \tilde{\nu} \rangle = \nu J \tilde{\nu}^t = 0$ , where  $J$  is the standard symplectic form).

Let us start to show that  $U(z) \neq 0$  iff  $z \in \partial\bar{\mathbb{D}}_g$ . Suppose  $U(z) \neq 0$ , it means that there is a  $z \in \bar{\mathbb{D}}_g$  such that  $\bar{w} = -zw$  for some  $w \in \mathbb{C}^g$ . Indeed, the above equation is satisfied by

$$z^{(k)} = \begin{pmatrix} \mathbb{I}_{g-k} & 0 \\ 0 & z_{(k)} \end{pmatrix}, \quad z_{(k)} \in \mathbb{D}_k \quad (\text{A.2})$$

and the  $g$ -dimensional vector  $w_{(k)} = (i\nu_1, \dots, i\nu_{g-k}, 0, \dots, 0)^t$ , with pure imaginary entries satisfies  $\bar{w}_{(k)} = -z^{(k)}w_{(k)}$  for  $k = 0, \dots, g$ . Notice that  $w_{(k)} \in \mathbb{C}^g$  corresponds to the real vector  $\nu_{(k)} = (0, \dots, 0; \nu_1, \dots, \nu_{g-k}, 0, \dots, 0)^t \in \mathbb{R}^{2g}$ , via the  $\nu$  map, where the first  $g$  entries are zero.

Moreover,

$$z^{(k)}\bar{z}^{(k)} - 1 = \begin{pmatrix} 0_{g-k} & 0 \\ 0 & z_{(k)}\bar{z}_{(k)} - \mathbb{I}_k \end{pmatrix}, \quad (\text{A.3})$$

is clearly a nonpositive matrix, therefore  $z^{(k)} \in \partial\bar{\mathbb{D}}_g$ . We see that  $\partial\bar{\mathbb{D}}_g$  decomposes into  $g$  components  $(\partial\bar{\mathbb{D}}_g)_k = \bar{\mathbb{D}}_k$ ,  $k = 0, \dots, g-1$ . Indeed, the boundaries related to the quotients of the Siegel spaces by arithmetic subgroups of  $Sp(2g, \mathbb{Z})$  turn out to be rational component of  $\partial(\bar{\mathbb{D}}_g)_k = \bar{\mathbb{D}}_k$ , similarly to the  $Sl(2, \mathbb{Z})$  case, where cusps are given by rational points of the boundary of  $\mathbb{H}_1$ , (see for example [27] for definitions and characterizations of the rational component of  $\partial(\bar{\mathbb{D}}_g)_k = \bar{\mathbb{D}}_k$ ).

Each boundary component  $(\partial\bar{\mathbb{D}}_g)_k$  is in correspondence to the real isotropic space in  $\mathbb{R}^{2g}$  spanned by the vectors  $\nu_{(k)} = (0, \dots, 0; \nu_1, \dots, \nu_{g-k}, 0, \dots, 0)^t$ , through the map

$$\Psi_z(\nu) = 2i\nu \begin{pmatrix} \mathbb{I} & -i\mathbb{I} \\ \mathbb{I} & i\mathbb{I} \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{I} \\ z \end{pmatrix}. \quad (\text{A.4})$$

Since under

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(2, \mathbb{R}), \quad (\text{A.5})$$

one has

$$\begin{pmatrix} \mathbb{I} \\ z \end{pmatrix} \rightarrow \begin{pmatrix} \mathbb{I} & -i\mathbb{I} \\ \mathbb{I} & i\mathbb{I} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \mathbb{I} & -i\mathbb{I} \\ \mathbb{I} & i\mathbb{I} \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{I} \\ z \end{pmatrix}, \quad (\text{A.6})$$

it follows that

$$\Psi_{g(z)}(\nu) = \Psi_z(\nu \cdot g). \quad (\text{A.7})$$

Whenever  $z \in \partial\bar{\mathbb{D}}_g$  is in the boundary of the closure of the  $g$ -dimensional disc,  $\nu \in \mathbb{R}^{2g}$  spans an isotropic subspace of  $\mathbb{R}^{2g}$ . Since isotropic spaces are stabilized by parabolic elements of  $Sp(2g, \mathbb{R})$  it follows from eq. (A.7) that parabolic subgroups stabilize the boundary. Moreover, the boundary of  $\bar{\mathbb{D}}_g$  decomposes into  $g$  components  $\partial\bar{\mathbb{D}}_g = \cup_{a=0}^{g-1} F_a$ , where

$$F_a = \begin{pmatrix} \mathbb{I}_{g-a} & 0 \\ 0 & z_{(a)} \end{pmatrix}, \quad z_{(a)} \in \mathbb{D}_a. \quad (\text{A.8})$$

$F_a$ ,  $a = 1, \dots, g-1$  are stabilized by parabolic subgroups  $Sp(2g, \mathbb{R})$  of corresponding dimensionality.

## A.2 $g = 1, 2$ examples

Let us start by applying the results of the previous section to the genus  $g = 1$  case in order to recover the parabolic subgroup  $\Gamma_\infty \subset Sl(2, \mathbb{Z})$  cusp stabilizer. Let  $z \in \bar{\mathbb{D}}_1$ , there is only one boundary component  $z = 1$  of dimension zero, (which is the image through the Cayley map of the  $\tau = i\infty$  cusp in  $\mathbb{H}_1$ ). Thus  $r = 0$ , and the isotropic subspace in  $\mathbb{R}^2$  related to the  $z = 1$  boundary is given by vectors of the form  $(0, \nu)$ . The  $Sl(2, \mathbb{R})$  equivariant map is given by

$$\Psi_z(\nu) = 2i\nu \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ z \end{pmatrix}, \quad (\text{A.9})$$

which under  $g \in Sl(2, \mathbb{R})$ ,  $\Psi_z(\nu \cdot g) = \Psi_{g(z)}(\nu)$ . Therefore a  $Sl(2, \mathbb{Z})$  modular transformation

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (\text{A.10})$$

stabilizes the  $z = 1$  cusp iff

$$(0, \nu) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (\nu c, \nu d) = (0, \nu). \quad (\text{A.11})$$

This leads to  $c = 0$ ,  $d = 1$ ,  $a = 1$ ,  $b \in \mathbb{Z}$ , and one recovers the well known  $g = 1$  parabolic<sup>1</sup> subgroup  $\Gamma_\infty \subset \Gamma$  of the modular group  $\Gamma \sim Sl(2, \mathbb{Z})$ , given by the unipotent matrices

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad b \in \mathbb{Z}. \quad (\text{A.12})$$

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<sup>1</sup>A matrix  $\gamma \in Sl(2, \mathbb{Z})$  is parabolic iff  $Tr(\gamma) = 2$ .

We consider now the  $g = 2$  case.  $\bar{\mathbb{D}}_2$  boundary has two components, the zero dimensional component  $F_0$

$$z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (\text{A.13})$$

and the one dimensional component  $F_1$

$$z = \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}, \quad z \in \bar{\mathbb{D}}_1. \quad (\text{A.14})$$

Let us start by obtaining the parabolic subgroup of  $Sp(4, \mathbb{Z})$  which stabilizes  $F_0$ . According to what explained in the previous section, the isotropic space in  $\mathbb{R}^4$  related to  $F_0$  is given by vectors of the form, ( $r = 0$ )

$$\nu = (0, 0, \nu_1, \nu_2). \quad (\text{A.15})$$

The parabolic subgroup  $P_0$  is obtained by the following condition

$$(0, 0, \nu_1, \nu_2) = (0, 0, \nu_1, \nu_2) \begin{pmatrix} a_1 & a_2 & b_1 & b_2 \\ a_4 & a_3 & b_4 & b_3 \\ c_1 & c_2 & d_1 & d_2 \\ c_4 & c_3 & d_4 & d_3 \end{pmatrix}, \quad (\text{A.16})$$

which implies that  $P_0$  is given by matrices in  $Sp(4, \mathbb{Z})$  of the form

$$\begin{pmatrix} \mathbb{I} & b \\ 0 & \mathbb{I} \end{pmatrix}, \quad (\text{A.17})$$

where  $b \in Mat(2, \mathbb{R})$  symmetric  $b = b^t$ .

In a similar way, the isotropic space related to  $F_1$  is

$$\nu = (0, 0, \nu_1, 0). \quad (\text{A.18})$$

The parabolic group  $P_1$  is thus determined by the condition

$$(0, 0, \nu_1, 0) = (0, 0, \nu_1, 0) \begin{pmatrix} a_1 & a_2 & b_1 & b_2 \\ a_4 & a_3 & b_4 & b_3 \\ c_1 & c_2 & d_1 & d_2 \\ c_4 & c_3 & d_4 & d_3 \end{pmatrix}, \quad (\text{A.19})$$

which gives  $c_1 = c_2 = d_2 = 0$ ,  $d_1 = 1$ . Imposing the  $Sp(4, \mathbb{Z})$  conditions gives (2.27).

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